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Estimation of total time on test transforms for stationary observations

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Abstract

By proving Chibisov–O'Reilly-type theorems for uniform empirical and quantile processes based on stationary observations, we establish a nonparametric large sample estimation theory for total time on test transforms. In particular, we obtain weak approximations for total time on test transforms also under the assumption of positively associated dependence, a kind of dependence that is encountered in many practical life testing situations. We derive similar asymptotic results for mixing sequences as well, another and often used structure of dependence for sequences.

Keywords: Total time on test; Life testing; Empirical processes; Quantile processes; Weighted metrics; Stationarity; Positive Association; Mixing

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1. Introduction

A statistical function that plays a central role in life testing of reliability is the so-called total time on test (TTT). Assume that n units are placed on test at time 0 and that successive failures are observed at times $X_{1:n} \leqslant \cdots \leqslant X_{n:n}$, the order statistics of the random variables $X_1, \ldots, X_n$ with common life distribution function F. Then, TTT up to the kth-order statistic, $T(X_{k:n})$, is defined by

$$T(X_{k:n}) = \sum_{i=1}^{k} (n+1-i)(X_{i:n} - X_{i-1:n}), \quad k = 1, \ldots, n,$$

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with $X_{0:n} \equiv 0$. In other words, $T(X_{k:n})$ represents the total test time of n simultaneously tested units accumulated up to the time of the kth failure. The statistics $T(X_{k:n})/n$ are called (unscaled) TTT-statistics and the plot of $T(X_{k:n})/n$ against k/n, $k = 1, \ldots, n$, is called the (unscaled) TTT plot. Correspondingly, this (unscaled) TTT plot can be constructed by the (unscaled) TTT function defined as

$$H_n(u) = \frac{1}{n} T(X_{[nu]:n}) = \frac{1}{n} \sum_{i=1}^{[nu]} (n + 1 - i)(X_{i:n} - X_{i-1:n}), \quad 0 \leqslant u \leqslant 1, \tag{1.1}$$

where for a nonnegative number x we denote by $[x]$ the smallest positive integer $\geqslant x$. Note that

$$H_n(1) = \frac{1}{n} \sum_{i=1}^{n} X_{i:n} = \bar{X}_n, \tag{1.2}$$

the sample mean.

The theoretical counterpart of $H_n(u)$, the (unscaled) TTT transform of F, is defined by

$$H_F(u) = \int_0^{Q(u)} (1 - F(x))\,\mathrm{d}x, \quad 0 \leqslant u \leqslant 1, \tag{1.3}$$

where $Q(u) = F^{-1}(u) = \inf\{x: F(x) \geqslant u\}, 0 < u \leqslant 1$, denotes the left-continuous quantile function. Since we assumed that the left end-point of the support of F is zero, we have $Q(0) = Q(0+) = 0$.

Let $F_n(x)$ and $Q_n(u)$ be the sample empirical distribution and quantile functions, respectively, defined by

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I\{X_i \leqslant x\},$$

$$0 \leqslant x < \infty \quad \text{and} \quad Q_n(u) = F_n^{-1}(u), \quad 0 \leqslant u \leqslant 1, \tag{1.4}$$

where $I(A)$ is the usual indicator function of the set A. In order to see that $H_F(u)$ is a natural theoretical counterpart of $H_n(u)$, it is easy to verify that for $(k-1)/n < u \leqslant k/n$ we have

$$\begin{aligned} \int_0^{Q_n(u)} (1 - F_n(x))\,\mathrm{d}x &= \int_0^{Q_n(k/n)} (1 - F_n(x))\,\mathrm{d}x \\ &= \sum_{i=1}^{k} \int_{X_{i-1:n}}^{X_{i:n}} (1 - F_n(x))\,\mathrm{d}x \\ &= \frac{1}{n} T(X_{k:n}) = H_n(u). \end{aligned} \tag{1.5}$$

In what follows, X denotes a generic random variable with distribution function F, and we assume throughout that the corresponding mean is finite:

$$\mu = EX = \int_0^\infty x \mathrm{d}F(x) = \int_0^\infty (1 - F(x)) \mathrm{d}x < \infty.$$

Then the scaled TTT function and transform are defined, respectively, by

$$D_n(u) = H_n(u)/H_n(1) = H_n(u)/\bar{X}_n, \quad 0 \leqslant u \leqslant 1, \tag{1.6}$$

$$D_F(u) = H_F(u)/H_F(1) = H_F(u)/\mu, \quad 0 \leqslant u \leqslant 1, \tag{1.7}$$

which are frequently more convenient in applications than their unscaled versions. For a study and use of the TTT transform, we refer to Bergman and Klefsjö (1984). This paper provides an enlightening overview of the scope of applications of the TTT transform.

The theory and applications of TTT when $\{X_n, n \geqslant 1\}$ is a sequence of independent, identically distributed (i.i.d.) random variables have been developed by many authors (cf., for example, Barlow and Campo, 1975; Barlow and Proschan, 1977; Bergman and Klefsjö, 1984; Csörgő et al. (CsCsH), 1986. In particular, the research monograph of CsCsH (1986) contains also a comprehensive review of developments concerning the asymptotic theory of TTT up to that time in the i.i.d. case, and provides the first general convergence theory for i.i.d.-based empirical total time on test and some related empirical reliability processes. However, in most reliability analyses and life testing situations, the basic sequence of observations $X_1, \ldots, X_n$ may not be independent. It is more realistic to assume some form of dependence among the components that are observed. One such notion of dependence that is especially applicable in reliability situations is that of positive dependence. As Barlow and Proschan (1975, p. 127) point out, in life testing circumstances: "This positive dependence among component life lengths arises from common environmental stresses and shocks, from components depending on common sources of power, and so on". A more specific example is the so-called shock model. This model is based on the assumption that the failures of the components are caused by different types of shocks striking single components or groups of components. For example, suppose that three independent sources of shocks are present in the environment. A shock from source 1 destroys component 1 at a random time U_1. A shock from source 2 destroys component 2 at a random time U_2. Finally, a shock from source 3 destroys both components at a random time U_{12}. Thus, the random life lengths of components 1 and 2 are $\min(U_1, U_{12})$ and $\min(U_2, U_{12})$, respectively (cf. Barlow and Proschan, 1975, Ch. 5). Consequently, the life distributions of these components are no longer independent. Rather, they are positively associated, which is a kind of dependence that is applicable to many situations encountered in practice.

A finite collection of random variables $X_1, \ldots, X_n$ is said to be positively associated (cf. Esary et al., 1967) if for any two coordinatewise nondecreasing functions $f, g: R^n \to R$,

$$\mathrm{Cov}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geqslant 0,$$

whenever this covariance is defined. An infinite family of random variables is positively associated if every finite subfamily is positively associated. An example of positively associated distribution is the joint distribution of the components of the mentioned shock model that can be derived as a multivariate exponential distribution (MVE). Let $X_1, \ldots, X_n$ be the random variables with MVE as their joint distribution. Since $X_1, \ldots, X_n$ can be represented as increasing functions of independent random variables, by property P_3 of positively associated random variables (cf. Esary et al., 1967), $X_1, \ldots, X_n$ are positively associated. For more applications of positive association in reliability theory, we refer to the book of Barlow and Proschan (1975).

In this paper we develop an asymptotic theory of the TTT function $H_n(u)$, first without assuming any specific structure of dependence, and then under the assumption of positively associated dependence. In particular, in addition to appropriate Glivenko–Cantelli-type theorems, we establish the weak convergence of the TTT empirical process $t_n(u)$ to a Gaussian process, as well as that of its called version $s_n(u)$. These two empirical processes are, respectively, defined by

$$t_n(u) = n^{1/2}(H_n(u) - H_F(u)), \quad 0 \leqslant u \leqslant 1 \tag{1.8}$$

and

$$s_n(u) = n^{1/2}(D_n(u) - D_F(u)), \quad 0 \leqslant u \leqslant 1. \tag{1.9}$$

Similar asymptotic results are also derived for stationary mixing sequences, another and often used structure of dependence for sequences. Our main technical tools are some Chibisov (1964) and O'Reilly (1974) type theorems that are established here for the uniform empirical and quantile processes of stationary sequences. For a review of weighted approximations of empirical and quantile processes in the i.i.d. case, we refer to Csörgő and Horváth (1993, Chrs. 4–6).

The structure of this paper is as follows. In Section 2 we define some basic notions and notations and present two basic theorems without using any specific structure of dependence. Then, still in Section 2, we apply these two basic theorems to obtain corresponding results for stationary positively associated sequences as well as for stationary mixing sequences. In Section 3 we study the problems of estimating the covariance structure of the limiting Gaussian processes. The above-mentioned Chibisov–O'Reilly-type theorems that are based on stationary sequences are stated and proved in Section 4. Finally, the proofs of the main results that are stated in Section 2 are carried out in Section 5.

2. Main results

We first introduce some notations that are necessary to state and establish asymptotic results for the TTT function $H_n(u)$. We assume throughout the paper that F is continuous. Then the quantile function Q of F is given by

$$Q(u) = F^{-1}(u) = \inf\{x\colon F(x) = u\}, \quad F(Q(u)) = u \in [0, 1]. \tag{2.1}$$

Hence, for $X_n \sim F$, $U_n = F(X_n)$ for all $n \geqslant 1$ are uniform-[0, 1] distributed random variables. The induced uniform empirical distribution function of $U_1, \ldots, U_n$ is defined by $E_n(u) = (1/n)\sum_{i=1}^{n} I(U_i \leqslant u) = F_n(Q(u))$, $0 \leqslant u \leqslant 1$, and its uniform empirical process α_n is given by

$$\{\alpha_n(u),\ 0 \leqslant u \leqslant 1\} = \{n^{1/2}(E_n(u) - u),\ 0 \leqslant u \leqslant 1\}. \tag{2.2}$$

The similarly induced uniform empirical quantile function is given by $G_n(u) = E_n^{-1}(u) = F(Q_n(u))$, $0 < u \leqslant 1$, $G_n(0) = G_n(0+)$, and its uniform quantile process u_n is defined by

$$\{u_n(u),\ 0 \leqslant u \leqslant 1\} = \{n^{1/2}(u - G_n(u)),\ 0 \leqslant u \leqslant 1\}. \tag{2.3}$$

When $\{U_n, n \geqslant 1\}$ is a stationary sequence of uniform-[0, 1] random variables, under certain conditions, we have the following weak convergence for α_n in $D[0, 1]$ with the Skorokhod J_1 topology (cf. Billingsley, 1968):

$$\alpha_n(\cdot) \xrightarrow{\mathscr{D}} B^*(\cdot) \quad \text{in } D[0, 1],$$

where $\{B^*(u),\ 0 \leqslant u \leqslant 1\}$ is a zero-mean Gaussian process with $B^*(0) = B^*(1) = 0$ and covariance function

$$\begin{aligned} EB^*(s)B^*(t) = s \wedge t - st & \\ &+ \sum_{k=2}^{\infty} \{\mathrm{Cov}(I(U_1 \leqslant s), I(U_k \leqslant t)) \\ &+ \mathrm{Cov}(I(U_k \leqslant s), I(U_1 \leqslant t))\}, \end{aligned} \tag{2.4}$$

where the series in (2.4) converges absolutely and $P\{B^*(\cdot) \in C[0, 1]\} = 1$. By stationarity we mean that the joint distribution of $U_{i+1}, \ldots, U_{i+m}$ does not depend on i for any fixed positive integer m. For the definitions of the spaces $C[0, 1]$ and $D[0, 1]$, we refer to Billingsley (1968, Chrs. 2 and 3), respectively.

Linking $H_n(u)$ with the uniform empirical and quantile functions, via (1.5) and (2.1), we have the following important conclusion:

$$\begin{aligned} H_n(u) &= \int_0^{F(Q_n(u))} (1 - F_n(Q(t)))\,\mathrm{d}Q(t) \\ &= \int_0^{G_n(u)} (1 - E_n(t))\,\mathrm{d}Q(t) \end{aligned} \tag{2.5}$$

for $0 \leqslant u \leqslant 1$ and each n.

Here, and throughout the paper, we use the convention $\int_a^b r\,\mathrm{d}l = \int_{[a,b)} r\,\mathrm{d}l$, $a < b$, for all occurring Lebesgue–Stieltjes integrals whenever r is a right-continuous and l is a left-continuous function.

Similarly, by (1.3) and (2.1), we have

$$H_F(u) = \int_0^u (1-t)\,\mathrm{d}Q(t), \quad 0 \leqslant u \leqslant 1. \tag{2.6}$$

The following theorem provides strong uniform consistency of $H_n(u)$ (its scaled version $D_n(u)$) to its theoretical counterpart $H_F(u)$ (its scaled version $D_F(u)$) under very general conditions. No specific structure of dependence is assumed.

Theorem 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables with common continuous distribution function F. Assume that $Q = F^{-1}$ is continuous on $[0, 1)$,*

$$\sup_{0 \leqslant x < \infty} |F_n(x) - F(x)| \xrightarrow{a.s} 0, \tag{2.7}$$

and $\{max\{X_n - \beta, 0\}, n \geqslant 1\}$ follows the strong law of large numbers (SLLN) *for any $\beta \geqslant 0$. Then we have*

$$\sup_{0 \leqslant u \leqslant 1} |H_n(u) - H_F(u)| \xrightarrow{a.s} 0 \tag{2.8}$$

and

$$\sup_{0 \leqslant u \leqslant 1} |D_n(u) - D_F(u)| \xrightarrow{a.s} 0. \tag{2.9}$$

Next we present weak convergence for the TTT empirical process t_n and its scaled version s_n. Again, no specific structure of dependence is assumed.

Theorem 2.2. *Let $\{X_n, n \geqslant 1\}$ be a stationary sequence of random variables with common distribution function F. Suppose that the density function $f = F'$ is continuous and positive on the open support of F,*

$$J = \sup_{0 < x < \infty} \frac{q(F(x))(1 - F(x))}{f(x)} = \sup_{0 < u < 1} \frac{q(u)(1-u)}{f(Q(u))} < \infty \tag{2.10}$$

and

$$\int_0^1 (1-u)^{\nu} (\log 1/(1-u))^{\beta}\,\mathrm{d}Q(u) < \infty, \tag{2.11}$$

where, for some $C > 0$, $0 < \nu \leqslant \frac{1}{2}$ and $\beta > \frac{1}{2}$, the continuous weight function q satisfies

$$q(u) \geqslant C(u(1-u))^{\nu}(\log 1/(u(1-u)))^{\beta} \quad \textit{for all } 0 < u < 1. \tag{2.12}$$

Then, as $n \to \infty$,

$$\alpha_n(\cdot)/q(\cdot) \xrightarrow{\mathscr{D}} B^*(\cdot)/q(\cdot) \quad \textit{in } D[0,1], \tag{2.13}$$

implies that

$$t_n(\cdot) \xrightarrow{\mathscr{D}} T_F^*(\cdot) \quad \textit{in } D[0,1] \tag{2.14}$$

and

$$s_n(\cdot) \xrightarrow{\mathscr{D}} S_F^*(\cdot) \quad \textit{in } D[0,1], \tag{2.15}$$

where $\{T_F^*(u), 0 \leqslant u \leqslant 1\}$ *and* $\{S_F^*(u), 0 \leqslant u \leqslant 1\}$ *are two Gaussian processes defined, respectively, by*

$$T_F^*(u) = -\int_0^u B^*(t)\,\mathrm{d}Q(t) - \frac{1-u}{f(Q(u))}B^*(u), \quad 0 \leqslant u \leqslant 1 \tag{2.16}$$

and

$$S_F^*(u) = \mu^{-1}T_F^*(u) - \mu^{-2}H_F(u)T_F^*(1), \quad 0 \leqslant u \leqslant 1. \tag{2.17}$$

Remark 2.1. Apart from our condition (2.12) for q, the form of condition (2.10) is the same that was used in CsCsH (1986) in proving (2.14) and (2.15), respectively, in the i.i.d. case (cf. their Theorems 6.2 and 7.2.). It is clearly associated with a notion of great importance in reliability theory, namely with that of the so-called failure or hazard rate function. Let the density function $f = F'$ be continuous and positive on the open support of F. Then the failure rate function r of F is defined by

$$r(x) = r_F(x) = \frac{f(x)}{1-F(x)}.$$

Condition (2.10), together with (2.12), implies that

$$r(x) \geqslant C(F(x)(1-F(x)))^{\nu}(\log 1/(F(x)(1-F(x))))^{\beta}/J \quad \text{for all } 0 < x < \infty.$$

In other words, $r(x)$ cannot be too close to zero near the two end points of the support of F. For example, when F is an increasing failure rate distribution (IFR), then (2.10) is satisfied if $r(0) > 0$, or $r(x) \geqslant K(F(x))^{\nu}$ for some $K > 0$ on a right-hand-side neighbourhood of 0. When F is a decreasing failure rate distribution (DFR), then (2.10) is satisfied if $r(\inf\{x\colon F(x) = 1\}) > 0$, or $r(x) \geqslant K(1-F(x))^{\nu}$ for some $K > 0$, on a left-hand-side neighbourhood of $\inf\{x\colon F(x) = 1\}$. Of course, (2.10) is satisfied if $r(x)$ has a "bathtub" shape, class of life distributions arising naturally in reliability situations. For a detailed analysis of the relationship of condition (2.10) to Chibisov–O'Reilly weight functions q, we refer to CsCsH (1986, Section 8).

Remark 2.2. Condition (2.11) is slightly stronger than the existence of the $(1/\nu)$th moment of X. Indeed, on extending the discussion in the Appendix of Hoeffding (1973), we see that (2.11) implies $EX^{1/\nu} < \infty$. This is not necessarily true conversely, but $EX^{1/\nu}(\log(1+X))^{(1+\beta)/\nu+\delta} < \infty$, with any $\delta > 0$, implies (2.11).

Remark 2.3. Sufficient conditions for (2.13) to hold true are given in Section 4, where we prove Chibisov–O'Reilly-type theorems for uniform empirical and quantile processes of stationary sequences.

Remark 2.4. The covariance functions of the limiting Gaussian processes (2.16) and (2.17) are computed and discussed in Section 3.

Theorems 2.1 and 2.2 enable us to establish an asymptotic theory for the TTT function $H_n(u)$ under the assumption of positively associated dependence.

Theorem 2.3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of positively associated random variables with common continuous distribution function F. Assume that $Q = F^{-1}$ is continuous on* $[0, 1)$ *and*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \operatorname{Cov}\left(X_n, \sum_{k=1}^{n} X_k\right) < \infty. \tag{2.18}$$

Then we have

$$\sup_{0 \leqslant u \leqslant 1} |H_n(u) - H_F(u)| \xrightarrow{a.s} 0$$

and

$$\sup_{0 \leqslant u \leqslant 1} |D_n(u) - D_F(u)| \xrightarrow{a.s} 0.$$

Theorem 2.4. *Let $\{X_n, n \geqslant 1\}$ be a stationary sequence of positively associated random variables with common distribution function F. Suppose that the density function $f = F'$ is continuous and positive on the open support of F. If*

$$\operatorname{Cov}(F(X_1), F(X_n)) = \mathrm{O}(n^{-\eta-\varepsilon}) \quad \textit{for some } \eta \geqslant (3 + \sqrt{33})/2 \textit{ and } \varepsilon > 0, \tag{2.19}$$

$$J = \sup_{0<u<1} \frac{q(u)(1-u)}{f(Q(u))} < \infty \quad \textit{and } EX^{2/(1-3/\eta)} < \infty,$$

where, for some $C > 0$, q satisfies

$$q(u) \geqslant C(u(1-u))^{(1-3/\eta)/2} \quad \textit{for all } 0 < u < 1,$$

then, as $n \to \infty$,

$$t_n(\cdot) \xrightarrow{\mathscr{D}} T_F^*(\cdot) \quad \textit{in } D[0,1]$$

and

$$s_n(\cdot) \xrightarrow{\mathscr{D}} S_F^*(\cdot) \quad \textit{in } D[0,1].$$

Remark 2.5. If the density function f is bounded on the open support of F, then (2.19) can be replaced by the sufficient condition

$$\operatorname{Cov}(X_1, X_n) = \mathrm{O}(n^{-\eta-\varepsilon}) \quad \text{for some } \eta \geqslant (3+\sqrt{33})/2 \text{ and } \varepsilon > 0.$$

To see this, we use the extended Hoeffding identity (cf. Yu, 1993, Theorem 2.3)

$$\operatorname{Cov}(f_1(X_1), f_2(X_2)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1'(x_1) f_2'(x_2) \operatorname{Cov}(I(X_1 \leqslant x_1), I(X_2 \leqslant x_2))\, dx_1\, dx_2, \tag{2.20}$$

for any absolutely continuous functions f_1 and f_2. Thus, from the above identity we obtain that

$$\mathrm{Cov}(F(X_1), F(X_n)) \leqslant \left(\sup_{0<x<\infty} f(x)\right)^2 \mathrm{Cov}(X_1, X_n),$$

which shows that (2.19) is satisfied by our sufficient condition.

We now turn to describing our asymptotic theory for the TTT function $H_n(u)$ under the assumption of mixing dependence. We first introduce the dependence notions of mixing that will be used.

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\mathscr{F}_1$ and $\mathscr{F}_2$ be two σ-algebras contained in $\mathscr{F}$. Define the following measures of dependence between $\mathscr{F}_1$ and $\mathscr{F}_2$:

$$\rho(\mathscr{F}_1, \mathscr{F}_2) = \sup_{X \in L_2(\mathscr{F}_1), Y \in L_2(\mathscr{F}_2)} \frac{|\mathrm{Cov}(X, Y)|}{(\mathrm{Var}\, X\, \mathrm{Var}\, Y)^{1/2}}$$

and

$$\alpha(\mathscr{F}_1, \mathscr{F}_2) = \sup_{A \in \mathscr{F}_1, B \in \mathscr{F}_2} |P(A \cap B) - P(A)P(B)|.$$

Let $\{X_n, n \geqslant 1\}$ be a sequence of real-valued random variables on $(\Omega, \mathscr{F}, P)$, $\mathscr{F}_n^m = \sigma(X_i, n \leqslant i \leqslant m)$ be σ-algebras generated by the indicated random variables, and put

$$\rho(n) = \sup_{k \geqslant 1} \rho(\mathscr{F}_1^k, \mathscr{F}_{n+k}^\infty) \quad \text{and} \quad \alpha(n) = \sup_{k \geqslant 1} \alpha(\mathscr{F}_1^k, \mathscr{F}_{n+k}^\infty).$$

The sequence $\{X_n, n \geqslant 1\}$ is said to be ρ-mixing, or α-mixing, according as $\rho(n) \to 0$, or $\alpha(n) \to 0$ as $n \to \infty$, respectively. It is well-known that $\alpha(n) \leqslant \rho(n)$.

Theorem 2.5. *Let $\{X_n, n \geqslant 1\}$ be a sequence of α-mixing random variables with common continuous distribution function F. Assume that $Q = F^{-1}$ is continuous on $[0, 1)$, $EX \log(1 + X) < \infty$, and*

$$\alpha(n) = O(n^{-\theta}) \quad \textit{for some } \theta > 0.$$

Then we have

$$\sup_{0 \leqslant u \leqslant 1} |H_n(u) - H_F(u)| \xrightarrow{a.s} 0$$

and

$$\sup_{0 \leqslant u \leqslant 1} |D_n(u) - D_F(u)| \xrightarrow{a.s} 0.$$

Theorem 2.6. *Let $\{X_n, n \geqslant 1\}$ be a stationary sequence of ρ-mixing random variables with common continuous distribution function F. Assume that $Q = F^{-1}$ is continuous on $[0, 1)$ and*

$$\sum_{n=1}^{\infty} \rho(2^n) < \infty.$$

Then we have

$$\sup_{0 \leqslant u \leqslant 1} |H_n(u) - H_F(u)| \xrightarrow{a.s} 0$$

and

$$\sup_{0 \leqslant u \leqslant 1} |D_n(u) - D_F(u)| \xrightarrow{a.s} 0.$$

Theorem 2.7. *Let $\{X_n, n \geqslant 1\}$ be a stationary sequence of α-mixing random variables with common distribution function F. Suppose that the density function $f = F'$ is continuous and positive on the open support of F. If*

$$\alpha(n) = \mathrm{O}(n^{-\theta-\varepsilon}) \quad \text{for some } \theta \geqslant 1 + \sqrt{2} \quad \text{and} \quad \varepsilon > 0, \tag{2.21}$$

$$J = \sup_{0<u<1} \frac{q(u)(1-u)}{f(Q(u)} < \infty \quad \text{and} \quad EX^{2/(1-1/\theta)} < \infty,$$

where, for some $C > 0$, q satisfies

$$q(u) \geqslant C(u(1-u))^{(1-1/\theta)/2} \quad \text{for all } 0 < u < 1,$$

then, as $n \to \infty$,

$$t_n(\cdot) \xrightarrow{\mathscr{D}} T_F^*(\cdot) \quad \text{in } D[0, 1]$$

and

$$s_n(\cdot) \xrightarrow{\mathscr{D}} S_F^*(\cdot) \quad \text{in } D[0, 1].$$

Theorem 2.8. *Let $\{X_n, n \geqslant 1\}$ be a stationary sequence of ρ-mixing random variables with common distribution function F. Suppose that the density function $f = F'$* is continuous and positive on the open support of F. If

$$\sum_{n=1}^{\infty} \rho(2^n) < \infty, \tag{2.22}$$

$$J = \sup_{0<u<1} \frac{q(u)(1-u)}{f(Q(u))} < \infty \quad \text{and} \quad EX^{2+\delta} < \infty \text{ for some } \delta > 0,$$

where for some $C > 0$ and $0 < \varepsilon < \delta$, q satisfies

$$q(u) \geqslant C(u(1-u))^{1/(2+\varepsilon)} \quad \text{for all } 0 < u < 1,$$

then as $n \to \infty$,

$$t_n(\cdot) \xrightarrow{\mathscr{D}} T_F^*(\cdot) \quad \text{in } D[0, 1]$$

and

$$s_n(\cdot) \xrightarrow{\mathscr{D}} S_F^*(\cdot) \quad \text{in } D[0, 1].$$

In particular, if

$$\sum_{n=1}^{\infty} \rho^{2/p}(2^n) < \infty \quad \textit{for some } p > 2, \tag{2.23}$$

$$J = \sup_{0<u<1} \frac{q(u)(1-u)}{f(Q(u))} < \infty \quad \textit{and} \quad EX^2 \log(1+X)^{2(1+\beta)+\delta} < \infty \ \textit{for some } \delta > 0,$$

where for some $C > 0$ *and* $\beta > \frac{1}{2}$, q *satisfies*

$$q(u) \geqslant C(u(1-u))^{1/2}(\log 1/(u(1-u)))^{\beta} \quad \textit{for all } 0 < u < 1,$$

then as $n \to \infty$,

$$t_n(\cdot) \xrightarrow{\mathscr{D}} T_F^*(\cdot) \quad \textit{in } D[0,1]$$

and

$$s_n(\cdot) \xrightarrow{\mathscr{D}} S_F^*(\cdot) \quad \textit{in } D[0,1].$$

3. On the problem of estimating the covariance structure of the limiting Gaussian processes

The aim of this section is to indicate and study the gap which would have to be filled in order to make statistical inference for TTT functions based on the results of Section 2. The essence of our problems along these lines is that the limiting processes T_F^* and S_F^* have a covariance structure that one would have to estimate in order to provide an initial basis for statistical inference. In order to initiate this line of thought, we now study our limiting Gaussian processes via their covariance functions.

First of all, the process $\{B^*(u), 0 \leqslant u \leqslant 1\}$ defined by (2.4) is no longer a standard Brownian bridge, due to the appearance of covariance among observations. It can be written as

$$B^*(u) = B(u) + B_d(u), \quad 0 \leqslant u \leqslant 1, \tag{3.1}$$

where $\{B(u), 0 \leqslant u \leqslant 1\}$ is a standard Brownian bridge, $\{B_d(u), 0 \leqslant u \leqslant 1\}$ is a mean-zero Gaussian process with covariance structure

$$\sigma(s,t) = \sum_{k=2}^{\infty} \{\mathrm{Cov}(I(U_1 \leqslant s), I(U_k \leqslant t)) + \mathrm{Cov}(I(U_k \leqslant s), I(U_1 \leqslant t))\}, \tag{3.2}$$

and $\{B(u), 0 \leqslant u \leqslant 1\}$ and $\{B_d(u), 0 \leqslant u \leqslant 1\}$ are two independent processes.

Note that the process $\{B_d(u), 0 \leqslant u \leqslant 1\}$ with the covariance structure (3.2) may not exist ($\sigma(t,t)$ may be negative). Hence, the main reason for the representation of (3.1) is to use it to compute the covariance function of the limiting process of $t_n(u)$ and that of its scaled version $s_n(u)$. Indeed, by combining (3.1) with (2.16) and (2.17), we obtain

$$T_F^*(u) = T_F(u) + T_d(u) \quad \text{and} \quad S_F^*(u) = S_F(u) + S_d(u), \qquad 0 \leqslant u \leqslant 1.$$

where

$$T_F(u) = -\int_0^u B(t)\,\mathrm{d}Q(t) - \frac{B(u)}{r(Q(u))}, \quad 0 \leqslant u \leqslant 1,$$

$$T_d(u) = -\int_0^u B_d(t)\,\mathrm{d}Q(t) - \frac{B(u)}{r(Q(u))}, \quad 0 \leqslant u \leqslant 1,$$

$$S_F(u) = \mu^{-1} T_F(u) - \mu^{-2} H_F(u)\, T_F(1), \quad 0 \leqslant u \leqslant 1,$$

and

$$S_d(u) = \mu^{-1} T_d(u) - \mu^{-2} H_F(u)\, T_d(1), \quad 0 \leqslant u \leqslant 1.$$

Clearly, $\{T_F(u), 0 \leqslant u \leqslant 1\}$ and $\{T_d(u), 0 \leqslant u \leqslant 1\}$ are two independent processes. So are also the processes $\{S_F(u), 0 \leqslant u \leqslant 1\}$ and $\{S_d(u), 0 \leqslant u \leqslant 1\}$. Hence, we have

$$ET_F^*(s)T_F^*(t) = ET_F(s)T_F(t) + ET_d(s)T_d(t) \tag{3.3}$$

and

$$ES_F^*(s)S_F^*(t) = ES_F(s)S_F(t) + ES_d(s)S_d(t). \tag{3.4}$$

Since $\{T_F(u), 0 \leqslant u \leqslant 1\}$ and $\{S_F(u), 0 \leqslant u \leqslant 1\}$ are the same mean-zero Gaussian processes as those obtained when the observations are i.i.d., one may want to utilize formulas obtained by CsCsH (1986, Section 8) in these calculations. However, we find that some terms are missed in the print of their expression for $ET_F(s)T_F(t)$ and, consequently, also in that for $ES_F(s)S_F(t)$. Here, based on the formula (2.20), we recalculated their forms and obtain that

$$\begin{aligned}
ET_F(s)T_F(t) &= \int_0^s\int_0^t EB(u)B(v)\,\mathrm{d}Q(u)\,\mathrm{d}Q(v) + \frac{1}{r(Q(s))}\frac{1}{r(Q(t))}EB(s)B(t)\\
&\quad + \frac{1}{r(Q(s))}\int_0^t EB(s)B(v)\,\mathrm{d}Q(v) + \frac{1}{r(Q(t))}\int_0^s EB(u)B(t)\,\mathrm{d}Q(u)\\
&= \int_0^{Q(s)}\int_0^{Q(t)} \mathrm{Cov}(I(X \leqslant x), I(X \leqslant y))\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + \frac{1}{r(Q(s))}\frac{1}{r(Q(t))}\mathrm{Cov}(I(F(X) > s), I(F(X) > t))\\
&\quad + \frac{1}{r(Q(s))}\int_0^1\int_0^{Q(t)} \mathrm{Cov}(I(I(F(X) > s) \leqslant x), I(X \leqslant y))\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + \frac{1}{r(Q(t))}\int_0^{Q(s)}\int_0^1 \mathrm{Cov}(I(X \leqslant x), I(I(F(X) > t) \leqslant y))\,\mathrm{d}x\,\mathrm{d}y\\
&= \mathrm{Cov}(K_F(X,s), K_F(X,t)),
\end{aligned} \tag{3.5}$$

where $K_F(X,u) = XI(X \leqslant Q(u)) + (Q(u) + 1/r(Q(u))I(X > Q(u))$. Similarly, we have

$$\begin{aligned} ES_F(s)S_F(t) &= \mu^{-2}ET_F(s)T_F(t) + \mu^{-4}H_F(s)H_F(t)ET_F(1)T_F(1) \\ &\quad - \mu^{-3}H_F(t)ET_F(s)T_F(1) - \mu^{-3}H_F(s)ET_F(t)T_F(1) \\ &= \mathrm{Cov}(L_F(X,s), L_F(X,t)), \end{aligned} \tag{3.6}$$

where $L_F(X,u) = \mu^{-1}K_F(X,u) - \mu^{-2}H_F(u)K_F(X,1) = \mu^{-1}K_F(X,u) - \mu^{-2}H_F(u)X$.

Just like in the i.i.d. case, the covariance functions of $\{T_d(u), 0 \leqslant u \leqslant 1\}$ and $\{S_d(u), 0 \leqslant u \leqslant 1\}$ are obtained by straightforward but very lengthy and tedious computations. Using the formula (2.20) again, we have

$$\begin{aligned} ET_d(s)T_d(t) &= \int_0^s \int_0^t \sigma(u,v)\,dQ(u)\,\mathrm{d}Q(v) + \frac{1}{r(Q(s))}\frac{1}{r(Q(t))}\sigma(s,t) \\ &\quad + \frac{1}{r(Q(s))}\int_0^t \sigma(s,v)\,\mathrm{d}Q(v) + \frac{1}{r(Q(t))}\int_0^s \sigma(u,t)\,\mathrm{d}Q(u) \\ &= \int_0^{Q(s)}\int_0^{Q(t)} \sigma(F(x),F(y))\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{r(Q(s))}\frac{1}{r(Q(t))}\sigma(s,t) \\ &\quad + \frac{1}{r(Q(s))}\int_0^{Q(t)} \sigma(s,F(y))\,\mathrm{d}y + \frac{1}{r(Q(t))}\int_0^{Q(s)} \sigma(F(x),t)\,\mathrm{d}x \\ &= \sum_{k=2}^{\infty} \{\mathrm{Cov}(K_F(X_1,s),K_F(X_k,t)) \\ &\quad + \mathrm{Cov}(K_F(X_k,s),K_F(X_1,t))\} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} ES_d(s)S_d(t) &= \mu^{-2}ET_d(s)T_d(t) + \mu^{-4}H_F(s)H_F(t)ET_d(1)T_d(1) \\ &\quad - \mu^{-3}H_F(t)ET_d(s)T_d(1) - \mu^{-3}H_F(s)ET_d(t)T_d(1) \\ &= \sum_{k=2}^{\infty} \{\mathrm{Cov}(L_F(X_1,s), L_F(X_k,t)) \\ &\quad + \{\mathrm{Cov}(L_F(X_k,s), L_F(X_1,t))\}. \end{aligned} \tag{3.8}$$

By (3.3)–(3.8), we obtain the following covariance functions for T_F^* and S_F^*

$$\begin{aligned} ET_F^*(s)T_F^*(t) &= \mathrm{Cov}(K_F(X,s),K_F(X,t)) \\ &\quad + \sum_{k=2}^{\infty} \{\mathrm{Cov}(K_F(X_1,s),K_F(X_k,t)) \\ &\quad + \mathrm{Cov}(K_F(X_k,s),K_F(X_1,t))\} \end{aligned} \tag{3.9}$$

and

$$ES_F^*(s)S_F^*(t) = \mathrm{Cov}(L_F(X,s), L_F(X,t)) + \sum_{k=2}^{\infty} \{\mathrm{Cov}(L_F(X_1,s), L_F(X_k,t)) + \mathrm{Cov}(L_F(X_k,s), L_F(X_1,t))\}. \tag{3.10}$$

In the light of these calculations we are now ready to describe a way to estimate the unknown quantities in our covariance functions. Define

$$\hat{t}_n(u) = n^{-1/2} \sum_{i=1}^{n} (K_F(X_i,u) - EK_F(X_i,u)), \quad 0 \leqslant u \leqslant 1$$

and

$$\hat{s}_n(u) = n^{-1/2} \sum_{i=1}^{n} (L_F(X_i,u) - EL_F(X_i,u)) = \mu^{-1}\hat{t}_n(u) - \mu^{-2}H_F(u)\hat{t}_n(1), \quad 0 \leqslant u \leqslant 1.$$

Then, by stationarity and (3.9) and (3.10), it is easy to see that

$$\lim_{n\to\infty} E\hat{t}_n(s)\hat{t}_n(t) = ET_F^*(s)T_F^*(t) \quad \text{and} \quad \lim_{n\to\infty} E\hat{s}_n(s)\hat{s}_n(t) = ES_F^*(s)S_F^*(t).$$

Hence $\{\hat{t}_n(u),\ 0 \leqslant u \leqslant 1\}$ and $\{\hat{s}_n(u),\ 0 \leqslant u \leqslant 1\}$ converge weakly to the same Gaussian processes that are the limits of $\{t_n(u),\ 0 \leqslant u \leqslant 1\}$ and $\{s_n(u),\ 0 \leqslant u \leqslant 1\}$, respectively, provided that $\{\hat{t}_n(u),\ 0 \leqslant u \leqslant 1\}$ converges weakly to a Gaussian process. Based on this finding, we propose here a resampling procedure (bootstrap) that may provide statistical inference for the process $\{t_n(u),\ 0 \leqslant u \leqslant 1\}$. Given a random sample $X_1, \ldots, X_n$, let $X_1^*, \ldots, X_n^*$ be a resample that is based on Künsch's (1989) moving block technique. As to the question whether the moving block technique provides consistent estimation for sample means and empirical processes in stationary case, we refer to Politis and Romano (1992), Shao and Yu (1993), and Bühlmann (1994).

Asymptotically, $X_1^*, \ldots, X_n^*$ have the known empirical distribution function F_n. Hence, the bootstrapped version of $\{t_n(u),\ 0 \leqslant u \leqslant 1\}$ can be written as

$$\hat{t}_n^*(u) = n^{-1/2} \sum_{i=1}^{n} (K_{F_n}(X_i^*,u) - E^*K_{F_n}(X_i^*,u)), \quad 0 \leqslant u \leqslant 1.$$

Unfortunately, the definition of the function K_F (cf. (3.5)) involves the unknown quantile-density function $(\mathrm{d}/\mathrm{d}u)Q(u) = 1/f(Q(u))$ that now has to be estimated. A direct estimator of $1/f(Q(u))$ can be found in Theorem 1 of Csörgő and Révész (1984) (cf. also Csörgö 1983, Theorem 4.1.3.) as

$$\frac{1}{f_n(Q_n(u))} = \frac{Q_n(u+a_n) - Q(u-a_n)}{2a_n},$$

where $a_n = n^{-\varepsilon}$ for some $0 < \varepsilon < \frac{1}{2}$. Hence, $K_{F_n}(X_1^*, u)$ should be computed as

$$X_i^* I(X_i^* \leqslant Q_n(u)) + \left(Q_n(u) + \frac{1-u}{f_n(Q_n(u))}\right) I(X_i^* > Q_n(u)).$$

We conjecture that under appropriate conditions $\{t_n^*(u), 0 \leqslant u \leqslant 1\}$ will converge weakly to $\{T_n^*(u), 0 \leqslant u \leqslant 1\}$ which, in turn, can be used to provide the foundations of statistical inference for the process $\{t_n(u), 0 \leqslant u \leqslant 1\}$. We note in passing that this procedure is different from the one suggested by CsCsH (1986) (cf. their Section 17).

4. Chibisov–O'Reilly-type theorems for uniform empirical and quantile processes of stationary sequences

As we mentioned before, our main technical tools are the Chibisov–O'Reilly-type theorems of this section for the uniform empirical and quantile processes of stationary sequences. The corresponding results for the uniform empirical process α_n under associated and mixing dependence assumptions are obtained by Shao and Yu (1996). Their results are based on the following basic theorem.

Theorem A. *We assume that q is continuous and positive on $(0, 1)$ and is non-decreasing in a neighbourhood of 0 and nonincreasing in a neighbourhood of 1. Let $\{U_n, n \geqslant 1\}$ be a stationary sequence of uniform-$[0, 1]$ random variables. Assume that for all $0 \leqslant s, t \leqslant 1$ and $n \geqslant 1$ we have*

(A1) $E|\alpha_n(t) - \alpha_n(s)|^p \leqslant C_1(t - s|^{p_1} + |t - s|^{r_1} n^{-p_2/2})$ *for some* $C_1 > 0, p > 2, p_1 > 1,$ $0 \leqslant r_1 \leqslant 1$ *and* $p_2 > 1 - r_1$,

(A2) $E(\alpha_n(t) - \alpha_n(s))^2 \leqslant C_2|t - s|^{r_2}$ *for some* $C_2 > 0$ *and* $0 > r_2 \leqslant 1$.

If, as $n \to \infty$, we have

$$\alpha_n(\cdot) \xrightarrow{\mathscr{D}} B^*(\cdot) \quad \text{in } D[0, 1], \tag{4.1}$$

then

$$\alpha_n(\cdot)/q(\cdot) \xrightarrow{\mathscr{D}} B^*(\cdot)/q(\cdot) \quad \text{in } D[0, 1], \tag{4.2}$$

where, for some $C > 0$ and $\beta > 1/2$, q satisfies

$$q(u) \geqslant C(u(1-u))^{v}(\log 1/(u(1-u)))^{\beta} \quad \text{for all } 0 < u < 1 \tag{4.3}$$

and

$$v = \min\left(\frac{p_1}{p}, \frac{r_1 + p_2}{p + p_2}, \frac{r_2}{2}\right). \tag{4.4}$$

The following corollary plays a crucial role in establishing many results of this paper. It describes the weighted tail behaviour of the uniform empirical process α_n. Its proof can be found in that of Theorem A in Shao and Yu (1996).

Corollary 4.1. *Under the assumption of Theorem A we have for any* $\varepsilon > 0$

$$\lim_{\theta\to 0}\limsup_{n\to\infty} P\left\{\sup_{0<u\leqslant\theta}|\alpha_n(u)|/q(u)\geqslant\varepsilon\right\} = \lim_{\theta\to 0} P\left\{\sup_{0<u\leqslant\theta}|B^*(u)|/q(u)\geqslant\varepsilon\right\} = 0$$

and

$$\lim_{\theta\to 0}\limsup_{n\to\infty} P\left\{\sup_{1-\theta\leqslant u<1}|\alpha_n(u)|/q(u)\geqslant\varepsilon\right\} = \lim_{\theta\to 0} P\left\{\sup_{1-\theta\leqslant u<1}|B^*(u)|/q(u)\geqslant\varepsilon\right\} = 0.$$

Remark 4.1. From (4.2), it is easy to obtain that

$$\sup_{0<u\leqslant\theta}|\alpha_n(u)|/q(u) \xrightarrow{\mathscr{D}} \sup_{0<u\leqslant\theta}|B^*(u)|/q(u)$$

and

$$\sup_{1-\theta\leqslant u<1}|\alpha_n(u)|/q(u) \xrightarrow{\mathscr{D}} \sup_{1-\theta\leqslant u<1}|B^*(u)|/q(u).$$

This implies that the results in Corollary 4.1 can be derived from (4.2) directly if

$$\lim_{\theta\to 0}\sup_{0<u\leqslant\theta}|B^*(u)|/q(u) = \lim_{\theta\to 0}\sup_{1-\theta\leqslant u<1}|B^*(u)|/q(u) \stackrel{\mathrm{P}}{=} 0.$$

The next corollary gives weak convergence for integral functionals of α_n, defined as

$$\Delta_n(u) = \int_0^u \alpha_n(t)\,\mathrm{d}Q(t), \quad 0\leqslant u\leqslant 1, \tag{4.5}$$

to their approximating Gaussian counterpart

$$\Delta(u) = \int_0^u B^*(t)\,\mathrm{d}Q(t), \quad 0\leqslant u\leqslant 1. \tag{4.6}$$

$\Delta_n(u)$ and $\Delta(u)$ will be used in establishing weak convergence of the TTT empirical processes t_n and s_n.

Corollary 4.2. *Under the conditions of Theorem A, if*

$$\int_0^1 (u(1-u))^{\nu}(\log 1/u(1-u)))^{\beta}\,\mathrm{d}Q(u) < \infty,$$

then

$$\Delta_n(\cdot) \xrightarrow{\mathscr{D}} \Delta(\cdot) \quad \text{in } D[0,1].$$

We now give a basic result on the Bahadur–Kiefer representation of quantiles, i.e., the deviation between empirical and quantile processes, based on stationary observations. The Chibisov–O'Reilly-type theorem for uniform quantile processes of stationary sequences can be easily derived from this representation (cf. Corollary 4.3). We first define the special weight function q^* by

$$q^*(u) = q^*(1-u) = u^{\nu}(\log 1/u)^{\beta}, \quad 0 < u \leqslant \tfrac{1}{2}, \tag{4.7}$$

for the same ν and β as defined in Theorem A, and the sequence of positive integers $\{k_n, n \geqslant 1\}$ by

$$k_n = k_n(\delta) = [\delta^{1-\nu} n^{(1-2\nu)/(2(1-\nu))} (\log n)^{\beta/(1-\nu)}], \tag{4.8}$$

where δ is a positive number that will be specified later on. From the definition of q^* and k_n, it is easy to verify that

$$\lim_{n\to\infty} \frac{k_n}{n^{1/2} q^*(k_n/n)} = (2(1-\nu))^{\beta}\delta. \tag{4.9}$$

This, together with the fact that $t/q^*(t)$ is increasing on $(0, \frac{1}{2}]$, implies that, for large n and $k_n/n \leqslant u \leqslant \frac{1}{2}$, we have

$$\frac{\delta}{2} \leqslant \frac{k_n}{n^{1/2} q^*(k_n/n)} \leqslant 2^{\beta}\delta, \tag{4.10}$$

$$\frac{\delta}{2} \leqslant \frac{k_n}{n^{1/2} q^*(k_n/n)} = n^{1/2} \frac{k_n/n}{q^*(k_n/n)} \leqslant n^{1/2} \frac{u}{q^*(u)}. \tag{4.11}$$

Put

$$\delta_n = \delta_n(\delta) = \frac{k_n}{n} = \max\{\delta^{1-\nu} n^{-1/(2(1-\nu))} (\log n)^{\beta/(1-\nu)}, 1/(n+1)\}. \tag{4.12}$$

Theorem 4.1. *Let $\{U_n, n \geqslant 1\}$ be a stationary sequence of uniform-$[0, 1]$ random variables. Then, under the same conditions as in Theorem A, we have for any $\delta > 0$*

$$\sup_{\delta_n \leqslant u \leqslant 1-\delta_n} |\alpha_n(u) - u_n(u)|/q(u) = O_p(1), \tag{4.13}$$

where q is the same function as defined in Theorem A.

Proof. By $u_n(k/n) = \alpha_n(U_{k:n})$, $k = 1, 2, \ldots, n$, we have

$$\sup_{0 \leqslant u \leqslant 1} |\alpha_n(u)| = \sup_{0 \leqslant u \leqslant 1} |u_n(u)|. \tag{4.14}$$

Using the definition of E_n and G_n, we obtain

$$\begin{aligned} u_n(u) &= n^{1/2}(E_n(G_n(u)) - G_n(u)) - n^{1/2}(E_n(G_n(u)) - u) \\ &= \alpha_n(G_n(u)) - n^{1/2}(E_n(G_n(u)) - u). \end{aligned} \tag{4.15}$$

It is well known that $0 \leqslant E_n(G_n(u)) - u \leqslant 1/n$, $u \in [0, 1]$. Hence, by (4.14) and (4.15), for any $\varepsilon > 0$ and $K > 0$, if n is large enough, we have

$$
\begin{aligned}
& P\left\{\sup_{0 \leqslant u \leqslant 1} |u_n(u) - \alpha_n(u)| \geqslant 2\varepsilon\right\} \\
& \quad \leqslant P\left\{\sup_{0 \leqslant u \leqslant 1} |\alpha_n(G_n(u)) - \alpha_n(u)| \geqslant \varepsilon\right\} \\
& \quad \leqslant P\left\{\sup_{0 \leqslant u \leqslant 1} |\alpha_n(G_n(u)) - \alpha_n(u)| \geqslant \varepsilon, |G_n(u) - u| < Kn^{-1/2}\right\} \\
& \qquad + P\left\{\sup_{0 \leqslant u \leqslant 1} |u_n(u)| \geqslant K\right\} \\
& \quad \leqslant P\{w(\alpha_n, Kn^{-1/2}) \geqslant \varepsilon\} + P\left\{\sup_{0 \leqslant u \leqslant 1} |\alpha_n(u)| \geqslant K\right\},
\end{aligned}
$$

where w is Lévy's modulus of continuity: $w(f, \delta) = \sup_{|s-t|<\delta} |f(s) - f(t)|$, $0 < \delta \leqslant 1$. Because α_n is tight under the condition (A1), the first term in the last inequality above converges to zero for any $\varepsilon > 0$. This, in turn, when combined with (4.1), yields

$$
\limsup_{n \to \infty} P\left\{\sup_{0 \leqslant u \leqslant 1} |u_n(u) - \alpha_n(u)| \geqslant 2\varepsilon\right\} \leqslant P\left\{\sup_{0 \leqslant u \leqslant 1} |B^*(u)| \geqslant K\right\}.
$$

Now, by letting $K \to \infty$, we conclude that

$$
\sup_{0 \leqslant u \leqslant 1} |\alpha_n(u) - u_n(u)| = \mathrm{O}_{\mathrm{p}}(1). \tag{4.16}
$$

By Corollary 4.1 and (4.16) combined, in order to prove (4.13), it is sufficient to prove that for any $\varepsilon > 0$

$$
\lim_{\theta \to 0} \limsup_{n \to \infty} P\left\{\sup_{\delta_n \leqslant u \leqslant \theta} |u_n(u)|/q(u) \geqslant \varepsilon\right\} = 0 \tag{4.17}
$$

and

$$
\lim_{\theta \to 0} \limsup_{n \to \infty} P\left\{\sup_{1-\theta \leqslant u \leqslant 1-\delta_n} |u_n(u)|/q(u) \geqslant \varepsilon\right\} = 0. \tag{4.18}
$$

By the definition of q, there exists $\theta_0 > 0$ such that q is nondecreasing on $(0, \theta_0]$. Then, for $i/n < u \leqslant (i+1)/n \leqslant \theta_0$, we have

$$
\begin{aligned}
\frac{|u_n(u)|}{q(u)} & \leqslant \max\left\{\frac{n^{1/2}|U_{i+1:n} - i/n|}{q(i/n)}, \frac{n^{1/2}|U_{i+1:n} - (i+1)/n|}{q(i/n)}\right\} \\
& \leqslant \frac{n^{1/2}|U_{i+1:n} - i/n|}{q(i/n)} + \frac{1}{n^{1/2}q(i/n)} \\
& \leqslant \frac{|\alpha_n(U_{i+1:n}-)|}{q(E_n(U_{i+1:n}-))} + \frac{1}{n^{1/2}q(i/n)},
\end{aligned}
$$

where $U_{1:n} \leqslant \cdots \leqslant U_{n:n}$ are the order statistics of $U_1, \ldots, U_n$. The above inequality yields

$$\sup_{\delta_n \leqslant u \leqslant \theta} \frac{|u_n(u)|}{q(u)} \leqslant \sup_{U_{k_n:n} \leqslant u \leqslant U_{\bar{k}_n:n}} \frac{|\alpha_n(u)|}{q(E_n(u))} + \frac{1}{n^{1/2}q(1/n)}, \tag{4.19}$$

where $\bar{k}_n$ is such that $(\bar{k}_n - 1)/n < \theta \leqslant \bar{k}_n/n$. On the other hand, since $E_n(u) \geqslant k_n/n = \delta_n$ for $u \geqslant U_{k_n:n}$ and q is nondecreasing on $(0, \theta_0]$, we have

$$\sup_{U_{k_n:n} \leqslant u \leqslant U_{\bar{k}_n:n}} \frac{|\alpha_n(u)|}{q(E_n(u))} \leqslant \sup_{0 < u < \delta_n} \frac{|\alpha_n(u)|}{q(u)} + \sup_{\delta_n \leqslant u \leqslant U_{\bar{k}_n:n}} \frac{|\alpha_n(u)|}{q(E_n(u))}.$$

This gives us

$$P\left\{\sup_{U_{k_n:n} \leqslant u \leqslant U_{\bar{k}_n:n}} \frac{|\alpha_n(u)|}{q(E_n(u))} \geqslant \varepsilon/2\right\}$$

$$\leqslant P\left\{\sup_{0 < u < \delta_n} \frac{|\alpha_n(u)|}{q(u)} \geqslant \varepsilon/4\right\} + P\left\{\sup_{\delta_n \leqslant u \leqslant U_{\bar{k}_n:n}} \frac{|\alpha_n(u)|}{q(E_n(u))} \geqslant \varepsilon/4\right\}. \tag{4.20}$$

By (4.11), $\sup_{\delta_n \leqslant u \leqslant 2\theta} |\alpha_n(u)|/q^*(u) \leqslant \delta/4$ implies that

$$E_n(u) \geqslant u - \delta q^*(u)/(4n^{1/2}) \geqslant u/2 \quad \text{for } \delta_n \leqslant u \leqslant 2\theta.$$

Also, $|\alpha_n(U_{\bar{k}_n:n})| \leqslant K$ implies

$$U_{\bar{k}_n:n} \leqslant \bar{k}_n/n + K/n^{1/2} \leqslant \theta + (K+1)/n^{1/2} \leqslant 2\theta$$

for $n \geqslant (K+1)^2/\theta^2$. Hence, for any $K > 0$, when $n \geqslant (K+1)^2/\theta^2$, we have

$$P\left\{\sup_{\delta_n \leqslant u \leqslant U_{\bar{k}_n:n}} \frac{|\alpha_n(u)|}{q(E_n(u))} \geqslant \varepsilon/4\right\}$$

$$\leqslant P\left\{\sup_{0 \leqslant u \leqslant 1} |\alpha_n(n)| > K\right\} + P\left\{\sup_{\delta_n \leqslant u \leqslant 2\theta} \frac{|\alpha_n(u)|}{q(E_n(u))} \geqslant \varepsilon/4\right\}$$

$$\leqslant P\left\{\sup_{0 \leqslant u \leqslant 1} |\alpha_n(n)| > K\right\} + P\left\{\sup_{\delta_n \leqslant u \leqslant 2\theta} \frac{|\alpha_n(u)|}{q^*(u)} > \delta/4\right\}$$

$$P\left\{\sup_{\delta_n \leqslant u \leqslant 2\theta} \frac{|\alpha_n(u)|}{q(u/2)} \geqslant \varepsilon/4\right\}. \tag{4.21}$$

Since $0 < v \leqslant \frac{1}{2}$ and $\beta > 0$, $n^{1/2}q(1/n) \to \infty$ as $n \to \infty$. Thus, by (4.1), (4.19)–(4.21), and Corollary 4.1, we obtain

$$\limsup_{\theta \to 0} \limsup_{n \to \infty} P\left\{\sup_{\delta_n \leqslant u \leqslant \theta} |u_n(u)|/q(u) \geqslant \varepsilon\right\} \leqslant P\left\{\sup_{0 \leqslant u \leqslant 1} |B^*(u)| \geqslant K\right\}$$

for each $K > 0$. Therefore, (4.17) is seen to be true by letting $K \to \infty$. Clearly, (4.18) can be proved in a similar way. This completes the proof of Theorem 4.1. □

The following corollary follows immediately from Theorems A and 4.1.

Corollary 4.3. *Let $\{U_n, n \geqslant 1\}$ be a stationary sequence of uniform-*$[0, 1]$ *random variables. Then, under the same conditions as in Theorem A, we have*

$$\hat{u}_n(\cdot)/q(\cdot) \xrightarrow{\mathscr{D}} B^*(\cdot)/q(\cdot) \quad \text{in } D[0,1],$$

where $\hat{u}_n(n) = u_n(u)I(\delta_n \leqslant u \leqslant 1 - \delta_n)$, and $q(\cdot)$ and $B^*(\cdot)$ are the weight function and the Gaussian process, respectively, as defined in Theorem A.

Remark 4.2. Since the special weight function $q^*(u)$ of (4.7) satisfies (4.3), all results in the section still hold after replacing $q(u)$ by $q^*(u)$. Hence, the monotonicity near the two end points of q in Theorem A can be dropped as long as (4.3) is satisfied. We note also that Theorem 4.1 with its δ_n as in (4.12) is an improved version of Theorem 2.1 of Csörgő and Yu (1996). This improved version is needed, for it plays a crucial role in the proof of our Theorem 2.2.

5. Proofs

Proof of Theorem 2.1. By (2.5) and (2.5), we have

$$\begin{aligned} &\sup_{0 \leqslant u \leqslant 1} |H_n(u) - H_F(u)| \qquad (5.1)\\ &\leqslant \sup_{0 \leqslant u \leqslant 1} \left| \int_0^{G_n(u)} (1 - E_n(t))\, dQ(t) - \int_0^{G_n(u)} (1 - t)\, dQ(t) \right| \\ &\quad + \sup_{0 \leqslant u \leqslant 1} |H_F(G_n(u)) - H_F(u)| \\ &\leqslant \sup_{0 \leqslant x < \infty} \left| \int_0^x (F_n(y) - F(y))\, dy \right| + \sup_{0 \leqslant u \leqslant 1} |H_F(G_n(u)) - H_F(u)|. \end{aligned}$$

Applying (2.7) and (4.14), we obtain

$$\sup_{0 \leqslant u \leqslant 1} |G_n(u) - u| = \sup_{0 \leqslant u \leqslant 1} |u - E_n(u)| = \sup_{0 \leqslant x < \infty} |F_n(x) - F(x)| \xrightarrow{a.s} 0.$$

Hence,

$$\sup_{0 \leqslant u \leqslant 1} |H_F(G_n(u)) - H_F(u)| \xrightarrow{a.s} 0,$$

Since the function H_F is uniformly continuous on $[0, 1]$, and thus (2.8) follows from (5.1) if we can show that

$$I_n = \sup_{0 \leqslant x < \infty} \left| \int_0^x (F_n(y) - F(y))\,\mathrm{d}y \right| \xrightarrow{a.s} 0. \tag{5.2}$$

Since we assume $EX = \int_0^\infty (1 - F(x))\,\mathrm{d}x < \infty$, for any $\varepsilon > 0$ we can choose $\beta > 0$ so large that

$$I^{(1)}(\beta) = \int_\beta^\infty (1 - F(x))\,\mathrm{d}x < \varepsilon/2.$$

Then

$$I_n \leqslant I^{(1)}(\beta) + I_n^{(2)}(\beta) + I_n^{(3)}(\beta),$$

where

$$I_n^{(2)}(\beta) = \frac{1}{n}\sum_{i=1}^n \int_\beta^\infty I(X_i > x)\,\mathrm{d}x = \frac{1}{n}\sum_{i=1}^n \max\{X_i - \beta, 0\},$$

$$I_n^{(3)}(\beta) = \sup_{0 \leqslant x \leqslant \beta} \left| \int_0^x (F_n(y) - F(y))\,\mathrm{d}y \right| \leqslant \beta \sup_{0 \leqslant x < \infty} |F_n(x) - F(x)| \xrightarrow{a.s} 0.$$

Thus, applying SLLN to $\{\max\{X_n - \beta, 0\}, n \geqslant 1\}$, we get

$$I_n^{(2)}(\beta) \xrightarrow{a.s} E\max\{X - \beta, 0\} = I^{(1)}(\beta).$$

Therefore, $\limsup_{n\to\infty} I_n \leqslant \varepsilon$, a.s. for all small ε, and this proves (2.8). Since $X_n = \max\{X_n - \beta, 0\}$ for $\beta = 0$, we have $\bar{X}_n \to \mu$, a.s., by SLLN, which proves (2.9) by (1.6) and (1.7). This completes our proof of Theorem 2.1. □

Proof of Theorem 2.2. By (2.2), (2.5) and (4.5), we can write

$$t_n(u) = -\int_0^{G_n(u)} \alpha_n(t)\,\mathrm{d}Q(t) - n^{1/2}\{H_F(u) - H_F(G_n(u))\} \tag{5.3}$$

$$= \{\Delta_n(u) - \Delta_n(G_n(u))\} - \Delta_n(u) - I_n(u), \quad 0 \leqslant u \leqslant 1,$$

where

$$I_n(u) = n^{1/2}\{H_F(u) - H_F(G_n(u))\}.$$

For any $K > 0$ and $\varepsilon > 0$, we have

$$P\left\{\sup_{0 \leqslant u \leqslant 1} |\Delta_n(u) - \Delta_n(G_n(u))| > \varepsilon\right\}$$

$$\leqslant P\left\{\sup_{0 \leqslant u \leqslant 1} |u_n(u)| > K\right\} + P\{w(\Delta_n, Kn^{-1/2}) \geqslant \varepsilon\}.$$

On the other hand, $\Delta_n(\cdot) \in C[0, 1]$, since Q is continuous. Hence, $\Delta_n(u)$ is tight by (2.13) and Theorem 8.2 of Billingsley (1968). Thus,

$$\limsup_{n\to\infty} P\left\{\sup_{0\leqslant u\leqslant 1} |\Delta_n(u) - \Delta_n(G_n(u))| > \varepsilon\right\} \leqslant P\left\{\sup_{0\leqslant u\leqslant 1} |B^*(u)| > K\right\}$$

follows by (2.13) and (4.14). Letting $K \to \infty$, we get

$$\sup_{0\leqslant u\leqslant 1} |\Delta_n(u) - \Delta_n(G_n(u))| \xrightarrow{P} 0. \tag{5.4}$$

Applying a one-term Taylor expansion and the fact that $\mathrm{d}H_F(u)/\mathrm{d}u = (1-u)/f(Q(u))$, we obtain

$$\begin{aligned} I_n(u) &= \frac{1-\tau_n(u)}{f(Q(\tau_n(u)))} u_n(u) \\ &= \left(\frac{1-\tau_n(u)}{f(Q(\tau_n(u)))} - \frac{1-u}{f(Q(u))}\right) u_n(u) + \frac{1-u}{f(Q(u))} u_n(u), \end{aligned} \tag{5.5}$$

where $\min\{u, G_n(u)\} \leqslant \tau_n(u) \leqslant \max\{u, G_n(u)\}$. The latter relation implied that $\tau_n(u)$ converges uniformly to u in probability on $[0, 1]$, since

$$\sup_{0\leqslant u\leqslant 1} |u - G_n(u)| = \sup_{0\leqslant u\leqslant 1} |E_n(u) - u| \xrightarrow{P} 0$$

by (2.13) and (4.14). Since the compound function $f(Q(\cdot))$ is uniformly continuous on $[\theta, 1-\theta]$ by assumption, and since $\sup_{0\leqslant u\leqslant 1} |u_n(u)| = \sup_{0\leqslant u\leqslant 1} |\alpha_n(u)|$ has a limit distribution, we have for any $0 < \theta < \frac{1}{4}$

$$\sup_{0\leqslant u\leqslant 1-\theta} \left|\frac{1-\tau_n(u)}{f(Q(\tau_n(u)))} - \frac{1-u}{f(Q(u))}\right| \sup_{0\leqslant u\leqslant 1} |u_n(u)| \xrightarrow{P} 0.$$

Hence, by (2.10)–(2.12), (4.14), (4.16), (5.3)–(5.5), Corollaries 4.1 and 4.2, and Theorem 4.2 of Billingsley (1968), (2.14) follows if we can show that for any $\varepsilon > 0$,

$$\lim_{\theta\to 0} \limsup_{n\to\infty} P\left\{\sup_{0<u\leqslant\theta} |I_n(u)| \geqslant \varepsilon\right\} = 0, \tag{5.6}$$

$$\lim_{\theta\to 0} \limsup_{n\to\infty} P\left\{\sup_{1-\theta\leqslant u<1} |I_n(u)| \geqslant \varepsilon\right\} = 0, \tag{5.7}$$

$$\lim_{\theta\to 0} P\left\{\sup_{0<u\leqslant\theta} \left|\frac{1-u}{f(Q(u))} B^*(u)\right| \geqslant \varepsilon\right\} = 0 \tag{5.8}$$

and

$$\lim_{\theta\to 0} P\left\{\sup_{1-\theta\leqslant u<1} \left|\frac{1-u}{f(Q(u))} B^*(u)\right| \geqslant \varepsilon\right\} = 0. \tag{5.9}$$

Since (2.10) and Corollary 4.1 already imply (5.8) and (5.9), we only need to prove (5.6) and (5.7). Without loss of generality, we assume that q is positive on $(0, 1)$, and is nondecreasing in a neighbourhood of 0 and nonincreasing in a neighbourhood of 1, for otherwise we could simply replace q by q^*. By (2.10), and (4.7)–(4.12), for any $\delta > 0$, we have

$$\sup_{0<u\leqslant\theta} |I_n(u)| = \sup_{0<u\leqslant\theta} n^{1/2}\left|\int_u^{G_n(u)} \frac{1-t}{f(Q(t))}\,\mathrm{d}t\right|$$

$$\leqslant \sup_{0<u<\delta_n} |I_n(u)| + \sup_{\delta_n\leqslant u\leqslant\theta} |I_n(u)|$$

$$\leqslant J\int_0^{\delta_n} \frac{n^{1/2}}{q(t)}\,\mathrm{d}t + J\int_0^{U_{k_n:n}} \frac{n^{1/2}}{q(t)}\,\mathrm{d}t + J\sup_{\delta_n\leqslant u\leqslant\theta}\left|\int_u^{G_n(u)} \frac{n^{1/2}}{q(t)}\,\mathrm{d}t\right|$$

$$\leqslant 2^{1+\nu}J\int_0^{\delta_n} \frac{n^{1/2}}{q^*(t)}\,\mathrm{d}t + J\int_{\delta_n}^{U_{k_n:n}} \frac{n^{1/2}}{q(t)} I\{U_{k_n:n} \geqslant \delta_n\}\,\mathrm{d}t$$

$$+ J\sup_{\delta_n\leqslant u\leqslant\theta} \frac{|u_n(u)|}{q(u)} + J\sup_{\delta_n\leqslant u\leqslant\theta} \frac{|u_n(u)|}{q(G_n(u))}$$

$$\leqslant \frac{2^{1+\nu+\beta}J\delta}{1-\nu} + 2J\sup_{\delta_n\leqslant u\leqslant\theta} \frac{|u_n(u)|}{q(u)} + J\sup_{\delta_n\leqslant u\leqslant\theta} \frac{|u_n(u)|}{q(G_n(u))},$$

where, by (4.10),

$$\int_0^{\delta_n} \frac{n^{1/2}}{q^*(t)}\,\mathrm{d}t \leqslant \frac{n^{1/2}}{(\log(1/\delta_n))^{\beta}}\int_0^{\delta_n} \frac{1}{t^{\nu}}\,\mathrm{d}t \leqslant \frac{n^{1/2}\delta_n}{(1-\nu)q^*(\delta_n)} \leqslant \frac{2^{\beta}\delta}{1-\nu}.$$

By (4.11), $\sup_{\delta_n\leqslant u\leqslant\theta}|u_n(u)|/q^*(u) \leqslant \delta/4$ implies

$$G_n(u) \geqslant u - \delta q^*(u)/(4n^{1/2}) \geqslant u/2 \quad \text{for } \delta_n \leqslant u \leqslant \theta.$$

Hence, by letting $\delta = (1-\nu)\varepsilon/(2^{3+\nu+\beta}J)$, we obtain

$$P\left\{\sup_{0<u\leqslant\theta} |I_n(u)| \geqslant \varepsilon\right\} \leqslant P\left\{\sup_{\delta_n\leqslant u\leqslant\theta} \frac{|u_n(u)|}{q(u)} \geqslant \frac{\varepsilon}{4J}\right\} + P\left\{\sup_{\delta_n\leqslant u\leqslant\theta} \frac{|u_n(u)|}{q(G_n(u))} \geqslant \frac{\varepsilon}{4J}\right\}$$

$$\leqslant P\left\{\sup_{\delta_n\leqslant u\leqslant\theta} \frac{|u_n(u)|}{q(u)} \geqslant \frac{\varepsilon}{4J}\right\} + P\left\{\sup_{\delta_n\leqslant u\leqslant\theta} \frac{|u_n(u)|}{q^*(u)} \geqslant \frac{\delta}{4}\right\}$$

$$+ P\left\{\sup_{\delta_n\leqslant u\leqslant\theta} \frac{|u_n(u)|}{q(u/2)} \geqslant \frac{\varepsilon}{4J}\right\},$$

and thus (5.6) follows from (4.17). The proof of (5.7) can be handled in a similar way. This proves (2.14).

Using the fact that $\mu = H_F(1)$, we can easily derive that

$$s_n(u) = \mu^{-1} t_n(u) - \mu^{-2} H_F(u) t_n(1) + t_n(u) \left\{ \frac{1}{H_n(1)} - \frac{1}{H_F(1)} \right\}$$

$$+ \frac{H_F(u)}{\mu} t_n(1) \left\{ \frac{1}{H_n(1)} - \frac{1}{H_F(1)} \right\}.$$

Thus, (2.15) follows from (2.14) if we show that

$$H_n(1) \xrightarrow{\mathrm{P}} H_F(1). \tag{5.10}$$

By (2.14), we have

$$t_n(1) = n^{1/2} \{ H_n(1) - H_F(1) \} \xrightarrow{\mathscr{D}} T(1).$$

This immediately implies (5.10). □

Proof of Theorem 2.3. By (2.18) and Theorem 2.1 of Yu (1993), (2.7) holds true. Hence, by Theorem 2.1, we complete proving our theorem if we can show that $\{\max\{X_n - \beta, 0\}, n \geqslant 1\}$ follows the SLLN for any $\beta \geqslant 0$. Note that $\max\{X - \beta, 0\}$ is an absolutely continuous and nondecreasing function of X with $EX = \int_\beta^\infty (1 - F(x))\,\mathrm{d}x$. Thus, $\{\max\{X_n - \beta, 0\}, n \geqslant 1\}$ is a sequence of positively associated random variables by property P_4 of positive association (cf. Esary et al., 1967). By (2.20),

$$\mathrm{Cov}(\max\{X_i - \beta, 0\}, \max\{X_j - \beta, 0\}) \leqslant \mathrm{Cov}(X_i, X_j) \quad \text{for all } i, j = 1, 2, \ldots$$

This shows that condition (2.18) is satisfied for the sequence $\{\max\{X_n - \beta, 0\}, n \geqslant 1\}$. Hence, by Birkel's (1989) SLLN, $\{\max\{X_n - \beta, 0\}, n \geqslant 1\}$ follows the SLLN and this also completes the proof of our theorem. □

Proof of Theorem 2.4. It follows easily by our basic theorem Theorem 2.2, and Theorem 2.4 and Corollary 2.4 of Shao and Yu (1996). □

Proof of Theorems 2.5 and 2.6. Assume that $\{X_n, n \geqslant 1\}$ is a sequence of mixing (α or ρ mixing) random variables. Then, for any real function g, $\{g(X_n), n \geqslant 1\}$ is still a sequence of mixing random variables with the same mixing decay rate ($\alpha(n)$ or $\rho(n)$). Hence, by Theorem 2.1, we can prove Theorems 2.5 and 2.6 if $\{X_n, n \geqslant 1\}$ follows the SLLN. In the case of α-mixing, this is true by Application 5 of Theorem 1 of Rio (1995), while in the case of ρ-mixing, it follows by Corollary 3.1 of Shao (1995). This proves Theorems 2.5 and 2.6 □

Proof of Theorems 2.7 and 2.8. They follow easily by our basic proposition, Theorem 2.2, combined with Theorem 2.2 and Corollary 2.2 of Shao and Yu (1996) for α-mixing, and with Theorem 2.3 and Corollary 2.3 of Shao and Yu (1996) for ρ-mixing, respectively. □

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